GENERAL ATOMIC DIVISION
GENERAL DYNAMICS CORPORATION
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GAMD-5910
Category A

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HYDRODYNAMIC FLOW EQUATIONS WITH A PLASTICITY RESISTANCE LAW

Sponsored by
Advanced Research Projects Agency
ARPA Order No. 71-62

Monitored by
Ballistic Research Laboratories
Contract No. DA-04-495-AMC-116(X)

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General Atomic Project 341

December 8, 1964

As the complete equations for plastic flow with arbitrarily large deformations are not readily available, the results of a number of scattered, relevant analyses have been assembled in the following pages to provide a summary of the appropriate equations. For simplification, tensor notation is used everywhere.

THE CONSERVATION LAWS

The equations of motion for a fluid are given below for a general tensor law of resistance:

$$\frac{D\rho}{Dt} + \rho\, u_{i,i} = 0, \qquad \text{continuity,} \qquad (1)$$

$$\rho \frac{Du_i}{Dt} = \sigma_{ij,j}, \qquad \text{momentum,} \qquad (2)$$

$$\rho \frac{D}{Dt}\left(E + \frac{1}{2} u_i u_i\right) = (\sigma_{ij} u_i)_{,j}, \qquad \text{energy,} \qquad (3)$$

where the convective derivative is given by the expression

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_i u_i \frac{\partial}{\partial x_i}$$

and repeated indices are summed. Differentiation with respect to a coordinate, x_i, is denoted by preceding the index with a comma, e.g.,

$$\frac{\partial}{\partial x_i} = (\quad)_{,i}.$$

The energy equation is simplified by elimination of the kinetic energy term in Eq. (3) by means of the momentum equation, resulting in

$$\rho \frac{DE}{Dt} = \sigma_{ij}\, u_{i,j}\,. \qquad (4)$$

The velocity gradient, $u_{i,j}$, can be written as the sum of a symmetric and an antisymmetric part. The symmetric part, ϵ_{ij}, is the strain-rate tensor and the antisymmetric part, ω_{ij}, is the angular velocity vector in tensor form.

Thus,

$$u_{i,j} = \epsilon_{ij} + \omega_{ij}, \tag{5}$$

where, by definition,

$$\epsilon_{ij} = \epsilon_{ji}, \qquad \omega_{ij} = -\omega_{ji} \tag{6}$$

The stress tensor is always symmetric, a result found by computing the total moment on an element of fluid and noting that the surface forces must cancel. Then,

$$\sigma_{ij} = \sigma_{ji}. \tag{7}$$

Interchanging dummy indices, using Eqs. (5) and (6), and noting that an expression equal to its negative is zero,

$$\sigma_{ij}\,\omega_{ij} = \sigma_{ji}\,\omega_{ji} = -\sigma_{ij}\,\omega_{ij} = 0.$$

The energy equation (4) then becomes

$$\rho\,\frac{DE}{Dt} = \sigma_{ij}\,\epsilon_{ij}. \tag{8}$$

The stress and strain tensors can be written as the sum of a hydrostatic part and a deviator part, defined as the total tensor less the hydrostatic part. The total tensors have been denoted above by Greek letters; the deviators will be denoted by Roman letters. Then the stress tensor becomes

$$\sigma_{ij} = -p\,\delta_{ij} + S_{ij}, \tag{9}$$

where

$$p = -\frac{1}{3}\,\sigma_{ii}, \qquad S_{ii} = 0, \tag{10}$$

and δ_{ij} is the Kronecker delta. Similarly, the strain rate tensor becomes

$$\epsilon_{ij} = \frac{1}{3}\,\theta\,\delta_{ij} + e_{ij}\,, \tag{11}$$

where

$$\theta = \epsilon_{ii}, \qquad e_{ii} = 0. \tag{12}$$

Using these relations, the five flow equations can be written in terms of the deviator quantities:

$$\frac{D\rho}{Dt} + \rho\theta = 0, \tag{13}$$

$$\rho \frac{Du_i}{Dt} = -p_{,i} + S_{ij,j} , \tag{14}$$

$$\rho \frac{DE}{Dt} + p\theta = S_{ij}\, e_{ij} . \tag{15}$$

THE CONSTITUTIVE EQUATIONS

A wide variety of material properties can be described by a constitutive equation which expresses the total strain rate as the sum of an elastic and a plastic part:

$$\underset{\text{(total)}}{e_{ij}} = \underset{\text{(elastic)}}{\frac{1}{2\mu}\dot{S}_{ij}} + \underset{\text{(plastic)}}{b\,S_{ij}} , \tag{16}$$

where μ is the shear modulus of elasticity; b is, in general, a function of stress and strain. Several special cases in which the second stress invariant

$$J_2 = S_{ij}\, S_{ij} \tag{17}$$

appears repeatedly are listed below. Where k appears, it denotes the yield strength of the material.

Classical viscosity

Shear modulus = $\mu = \infty$; $b = 1/2\,\bar{\mu}$, $\bar{\mu}$ = dynamic viscosity; (18)

Prandtl-Reuss plasticity equations

$$\left.\begin{array}{ll} b = e_{ij}\, S_{ij}/k^2 & \text{for } J_2 = k^2, \\ b = 0 & \text{for } J_2 < k^2; \end{array}\right\} \tag{19}$$

Perznya plasticity equations

$$\left.\begin{aligned} b &= \frac{\gamma}{\sqrt{J_2}} \left(\frac{\sqrt{J_2}}{k} - 1 \right)^{\delta} & \text{for } J_2 \geq k^2, \\ b &= 0 & \text{for } J_2 < k^2; \end{aligned}\right\} \qquad (20)$$

Classical elasticity

$$b = 0; \qquad (21)$$

Rigid-plastic equations

$$\mu = \infty, \text{ with Eq. (19) or Eq. (20).} \qquad (22)$$

MATERIAL ROTATION

The general constitutive equation, (16), is only valid for coordinates fixed in the material. If it is to be used in connection with a set of Eulerian flow equations, for which the coordinate axes are fixed in space, a correction for rotation of the material must be included. The physical basis for this correction term is that when the material rotates there is a change in the components of the stress tensor, even though the physical state of stress may not change. Let $\overline{T}_{ij}$ denote the components of any tensor in material axes and T_{ij} the components in fixed axes. Then

$$T_{ij} = \overline{T}_{k\ell}\, a_{ik}\, a_{j\ell} \qquad (23)$$

$$\overline{T}_{ij} = T_{k\ell}\, a_{ki}\, a_{\ell j} \qquad (24)$$

and the direction cosines, which are time-dependent, satisfy the identities

$$a_{ij}\, a_{ik} = \delta_{jk}, \qquad (25)$$

$$a_{ji}\, a_{ki} = \delta_{jk}. \qquad (26)$$

Equation (16) becomes, in this notation,

$$\overline{e}_{ij} = \frac{1}{2\mu}\, \dot{\overline{S}}_{ij} + b\, \overline{S}_{ij}. \qquad (27)$$

It is required to replace the first term on the right with appropriate fixed-axis derivatives. Differentiating Eq. (24)

$$\dot{\bar{T}}_{ij} = \dot{T}_{k\ell}\, a_{ki}\, a_{\ell j} + T_{k\ell}\, \dot{a}_{ki}\, a_{\ell j} + T_{k\ell}\, a_{ki}\, \dot{a}_{\ell j}$$

and multiplying by $a_{pi}\, a_{qj}$,

$$\dot{\bar{T}}_{ij}\, a_{pi}\, a_{qj} = \dot{T}_{pq} + T_{kq}\, \dot{a}_{ki}\, a_{pi} + T_{p\ell}\, a_{qj}\, a_{\ell j}. \tag{28}$$

The quantities $-a_{ik}\, \dot{a}_{jk}$ are the components of the angular velocity tensor, ω_{ij}. To see this, we make use of the angular velocity vector, $\vec{\Omega}$, about which the material axes are rotating. Let $\vec{a}_\ell$ denote the unit vector along the ℓ axis in the rotating material. The angular velocity of the endpoint of this vector is in vector notation

$$\vec{a} = \vec{\Omega} \times \vec{a}_\ell$$

or in tensor notation

$$\dot{a}_{r\ell} = \epsilon_{rjk}\, \Omega_j\, a_{k\ell}, \tag{29}$$

where

$$\epsilon_{ijk} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix} \text{ according as } i, j, k \begin{Bmatrix} \text{form an even} \\ \text{form an odd} \\ \text{do not form a} \end{Bmatrix} \begin{matrix} \text{permutation} \\ \text{of } 1,2,3. \end{matrix} \tag{30}$$

Multiplying Eq. (29) by $a_{i\ell}$ and using Eq. (26),

$$\dot{a}_{r\ell}\, a_{i\ell} = \epsilon_{rji}\, \Omega_j.$$

The angular velocity vector, $\vec{\Omega}_j$, is related to the angular velocity tensor by

$$\omega_{ik} = \epsilon_{ijk}\, \Omega_j. \tag{31}$$

Combining these two equations and changing indices,

$$\dot{a}_{r\ell}\, a_{i\ell} = -\omega_{ri} \tag{32}$$

and Eq. (28) becomes

$$\dot{\bar{T}}_{ij}\, a_{pi}\, a_{qj} = T_{pq} + T_{kq}\, \omega_{kp} + T_{p\ell}\, \omega_{\ell q}. \tag{33}$$

Multiplying Eq. (27) by $a_{pi}\ a_{qj}$, eliminating the barred quantities by means of Eq. (33), and putting the deviator stress tensor, S_{ij}, for the general tensor, T_{ij}, we get, finally,

$$e_{ij} = \frac{1}{2\mu}\left(\dot{S}_{ij} + S_{kj}\ \omega_{ki} + S_{ik}\ \omega_{kj}\right) + b\ S_{ij}. \tag{34}$$

The term on the right of the energy equation, (15), can be obtained by multiplying Eq. (34) by S_{ij} and using the expression for J_2 in Eq. (17). The terms involving ω_{ij} disappear by symmetry considerations, as outlined in the equations below:

$$S_{ir}\ \omega_{rj}\ S_{ij} = -\ S_{ir}\ \omega_{jr}\ S_{ij} = -\ S_{ij}\ \omega_{rj}\ S_{ir} = 0.$$

The Lagrangian time derivative in Eq. (34) should be replaced by the convective derivative in the Eulerian formulation; the energy equation, in its final form, then becomes

$$\rho\,\frac{DE}{Dt} + p\ \theta = \frac{1}{4\mu}\ \frac{DJ_2}{Dt} + b\ J_2. \tag{35}$$

The first term on the right is the rate of increase of distortional strain energy and the second term is the work done by the plastic strains. The strain energy is recoverable, but the work done by plastic strains is not.

THE COMPLETE EQUATIONS

For convenience, the equations for plastic flow are assembled below and include the equation of state, which has not been introduced before.

Continuity Equation

$$\frac{D\rho}{Dt} + \rho\ \theta = 0, \tag{36}$$

Momentum Equation

$$\rho\,\frac{Du_i}{Dt} = -\ p_{,i} + S_{ij,j}, \tag{37}$$

Energy Equation

$$\rho\,\frac{DE}{Dt} + p\ \theta = e_{ij}\ S_{ij} = \frac{1}{4\mu}\ \frac{DJ_2}{Dt} + b\ J_2, \tag{38}$$

Equation of State

$$p = F(E, \rho). \tag{39}$$

Constitutive Equations

$$e_{ij} = \frac{1}{2\mu}\left(\frac{DS_{ij}}{Dt} + S_{kj}\,\omega_{ki} + S_{ik}\,\omega_{kj}\right) + b\,S_{ij}, \tag{40}$$

$$S_{ii} = 0, \tag{41}$$

where

$$\left.\begin{array}{ll} b = e_{ij}\,S_{ij}/k^2 & \text{for } J_2 = k^2, \\ b = 0 & \text{for } J_2 < k^2. \end{array}\right\} \tag{42}$$

Defining Equations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \frac{1}{3}\theta\,\delta_{ij} \qquad \text{(deviator strain rate)}, \tag{43}$$

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \qquad \text{(rotation rate)}, \tag{44}$$

$$J_2 = S_{ij}\,S_{ij}, \tag{45}$$

$$\theta = u_{i,i}. \tag{46}$$

THE PLASTIC-RIGID EQUATIONS

This situation is considerably simplified if the plastic-rigid assumption is introduced, which is, in effect, to neglect the elastic regime. Then the constitutive equation, (40), becomes

$$e_{ij} = b\,S_{ij}, \tag{47}$$

and Eq. (42) is simply

$$b = e_{ij}\,S_{ij}/k^2 = \frac{1}{k}(e_{ij}\,e_{ij})^{1/2}. \tag{48}$$

Differentiating Eq. (47), putting $\frac{1}{2b} = \bar{\mu}$ (denoting a viscositylike term), and making some straightforward manipulations,

$$S_{ij,j} = \bar{\mu} \left(\nabla^2 u_i + \frac{1}{3} \theta_{,i} \right) + 2 e_{ij} \bar{\mu}_{,j}. \qquad (49)$$

The equations are similar to the Navier-Stokes equations, but the "viscosity," $\bar{\mu}$, is not a constant, as in the usual viscous flow. Summarizing these results, we have for plastic-rigid flow six equations in the six unknowns, ρ, u_i, p, and E. The last three equations for the variables $\bar{\mu}$, e_{ij}, and θ are not essential and are not counted in the sum.

$$\frac{D\rho}{Dt} + \rho \theta = 0, \qquad (50)$$

$$\rho \frac{Du_i}{Dt} = - p_{,i} + \bar{\mu} \left(\nabla^2 u_i + \frac{1}{3} \theta_{,i} \right) + 2 e_{ij} \bar{\mu}_{,j}, \qquad (51)$$

$$\rho \frac{DE}{Dt} + p \theta = k^2/2\bar{\mu} = k \sqrt{e_{ij} e_{ij}} , \qquad (52)$$

where

$$p = F (E,\rho), \qquad (53)$$

$$\bar{\mu} = \frac{k}{2\sqrt{e_{ij} e_{ij}}} , \qquad (54)$$

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) - \frac{1}{3} \theta \delta_{ij}, \qquad (55)$$

$$\theta = u_{i,i}. \qquad (56)$$

CYLINDRICAL COORDINATES

The equations for axially symmetric flows are given below. u and w denote the radial and axial velocity components. r and z denote coordinates in the radial and axial directions. Dots denote ordinary time derivatives and subscripts denote differentiation with respect to

the indicated coordinate.

$$\dot{\rho} + u\rho_r + w\rho_z + \rho\theta = 0,$$

$$\rho\,(\dot{u} + uu_r + wu_z) = -p_r + \bar{\mu}\,(u_{rr} + \frac{1}{r}\,u_r + u_{zz} + \frac{1}{3}\,\theta_r)$$

$$+ 2e_{11}\,\bar{\mu}_r + 2e_{13}\,\bar{\mu}_z,$$

$$\rho\,(\dot{w} + uw_r + ww_z) = -p_z + \bar{\mu}\,(u_{rr} + \frac{1}{r}\,w_r + w_{zz} + \frac{1}{3}\,\theta_z)$$

$$+ 2e_{13}\,\bar{\mu}_r + e_{33}\,\bar{\mu}_z,$$

$$\rho\,(\dot{E} + uE_r + wE_z) + p\theta = k\sqrt{e_{ij}\,e_{ij}}\;,$$

$$e_{11} = u_r - \frac{\theta}{3}\,,$$

$$e_{31} = e_{13} = \frac{1}{2}\,(u_z + w_r) - \frac{\theta}{3}\,,$$

$$e_{33} = w_z - \frac{\theta}{3}\,,$$

$$\theta = u_r + \frac{u}{r} + w_z\,,$$

$$e_{ij}\,e_{ij} = u_r^2 + \frac{1}{2}\,(u_z + w_r)^2 + w_z^2 + (\frac{u}{r})^2 - \frac{\theta}{3}^2\,,$$

$$\bar{\mu} = k/2\,\sqrt{e_{ij}\,e_{ij}}\,, \qquad p = F(E,\rho).$$

SPHERICALLY SYMMETRIC FLOWS

In the case of spherically symmetric flows, the equations are the following, in which u denotes radial velocity and r denotes distance from the origin.

$$\dot{\rho} + u\rho_r + \rho\theta = 0\,,$$

$$\rho\,(\dot{u} + uu_r) = -p_r + \bar{\mu}\,(u_{rr} + \frac{2\,u_r}{r}\,) + 2e_{11}\,\bar{\mu}_r\,,$$

$$\rho\,(\dot{E} + u\,E_r) + p\,\theta = k\sqrt{\frac{2}{3}}\;\left| u_r - \frac{u}{r} \right| ,$$

$$e_{11} = u_r - \theta/3 , \qquad \theta = u_r + 2\,\frac{u}{r} ,$$

$$\bar{\mu} = \sqrt{\frac{3}{8}}\;\frac{k}{\left| u_r - \frac{u}{r} \right|} .$$

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